The paraheight of linear groups

Behnam Razzaghmaneshi[1]

[1] Assistant Professor of Mathematics, Algebra, Islamic Azad University, Talesh Branch, Talesh, Iran

Correspondence: Behnam Razzaghmanesh, Assistant Professor of Mathematics, Algebra, Islamic Azad University, Talesh Branch, Talesh, Iran. E-mail: b_razzagh@yahoo.com

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Abstract

If G is a subgroup of GL (n, F) G has paraheight at most w + [log, n!]. If G is a subgroup of GL (n, R) where R is a finitely generated integral domain then G has finite paraheight.

Keywords: linear groups, paraheight groups, finite groups.

A altura paralela dos grupos lineares

Resumo

Se G é um subgrupo de GL (n, F) G tem altura paralela no máximo w + [log, n!]. Se G é um subgrupo de GL (n, R) onde R é um domínio integral finitamente gerado, então G tem altura paralela finita.

Palavras-chave: grupos lineares, grupos de altura paralela, grupos finitos.

1. Introduction

Let G be an arbitrary group A a G-module and H the split extension of A by G. We say that A is a hypercyclic G-module with paraheight y if A is H-hypercyclic (as a normal subgroup of H) with H-paraheight y.

1.1 Proposition

Let G be a finite group and A a hypercyclic G-module. Then A has paraheight at most W. If A is torsion-free as Z-module A has paraheight at most 1 G 1 and if A has jinite exponent e as Z-module and (e, 1 G 1) = 1 then A has paraheight at most 1 G 1 [logs e].

Proof. 1

Suppose that A is Z-torsion-free. Embed A in V = Q 0 ~ A by identifying A and 1 @ A. V is a QG-module containing an ascending series of QG-submodules whose factors have Q-dimension at most 1. Also, V is completely reducible as QG-module (Extension of Maschke’s Theorem, Ref. [7], IV.8.j) and thus V is a direct sum of irreducible QG-modules of Q-dimension 1. Let Vi be the homogeneous components of V as QG-module and put Ai = A n Vi. Suppose that Ai i: (0). If g E G there exists a rational number s such that ag = sa for all a in Ai. Since A is hypercyclic there exists b EA\ (O) such that (b) is a G-submodule of Ai, that is 6g = nb for some integer n. Hence s = n is an integer (since A is torsion-free) and consequently every Z-submodule of Ai is a G-module.

Thus, A has paraheight at most Y and clearly Y < / G j. 2. Suppose that A is an elementary abelian p-group where (p, / G I) = 1. By the extension of Maschke’s Theorem A is completely reducible as F, G-module. Let A, A,..., A, be the homogeneous components of A as lF,G-module. Since A is hypercyclic G acts on each Ai as a group of scalars and thus every Z-submodule of each Ai is a G-submodule. Therefore, A has paraheight at most Y < 1 G I. If A has finite exponent e then A contains a characteristic series of length at most [log, e] whose factors are elementary abelian. Hence by 2. if (e, 1G I) = 1, then A has paraheight at most I G / [log, e].

2. General Case

A contains a free abelian subgroup E such that A/E is a periodic abelian group. Then E1 = fiBEG Eg is a free abelian G-submodule of A and A/E, is still periodic as Z-module (since G is finite).

Hence by 1. Above we may assume that A is periodic as Z-module. A = 0, A, where A, is the p-primary component of A. If I3 is a G-submodule of A such that every Z-submodule of every primary component of B is G-invariant, then every Z-submodule of B is G-invariant. Hence the paraheight of A is equal to the upper bound of the paraheights of the Ahus, we may suppose that A is a p-group.

Let A, be the G-submodule of A generated by all the irreducible G-sub-modules of A of order p and inductively define A,+, / A, = (A/A,),. Put A, = (JB A A, +JA, is elementary abelian and completely reducible as lF, G-module. There exists only a finite number (r say) of irreducible lF, G-modules of order p up to isomorphism. Let A, +, (j)/A, j = 1, 2, ..., r, be the homogeneous components A, +JA, as IF, G-module. As in 2. Above every Z-submodule of A, +, (j)/A, is G-invariant and thus A, has paraheight at most w.

It remains to show that A, = A. If a E A, (aG) is a finite G-submodule of A (since G is finite and A is periodic). Now A is hypercyclic, so there exists a finite series of G-submodules, {0} = B, C B, C **. C B, = (aG) such that (B,: B, -J = p f or each i. Clearly Bi C Ai for each i and thus (aG) C A. Therefor A = A, and 4.1 is proved.

It follows from 4.1 and 3.3 that a locally supersoluble linear group has paraheight less than ~2 and that a locally supersoluble linear group over a finitely generated integral domain is parasoluble (use the existence of a nilpotent subgroup of finite index and for the second part [lo, 4.101). TO obtain the relative case and the bounds in terms of 11we have to a little more work that remarkably parallels the hypercentral situation ([lo] Chap. 8).

2.1 LEMMA

Let V be a vector space of dimension n over the field F, G a subgroup of AutF (V) and W an FG-submodule of V of F-dimension d. If A is a G-hypercyclic normal subgroup of G stabilizing the series (0) C W 2 V then A has G-paraheight at most d (n - d).

2.2 Proof

If a E A let a’ be the linear mapping of VI W into W given by v + W H v (u - 1). The map 4: a ++ a’ is a (group) monomorphism of A into the additive group of Hom (V/W, W). The latter is an FG-module, the G action being given byf 9: v + W H (vg-l + W) fg; f E Hom (V/W, W) and g E G. A is abelian and so is a G-module via conjugation. A simple check shows that $ is a G-module homomorphism.

Let B be the F-subspace of Hom, (V/ W, W) spanned by A+. B is an FG-module and since A is G-hypercyclic contains an ascending series of FG-submodules whose factors have F-dimension at most 1. B has F-dimension at most d (n - d). Thus, there exists a series of FG-submodules (0) = B, C B, C ... C B, = B where each B, +lIB, has F-dimension 1 and r < d (n - d).

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Since A$ is G-hypercyclic and spans B there exists c in B, +, /B, \{O} such that (c) is a G-module. Now if g E G there exists 01 in F such that xg = OLX for all x in B, +, / B. Hence EC = nc for some integer n and thus a = TZl. Therefore, every Z-submodule of B, +, /B, is a G-submodule and so B has paraheight (as G-module) at most Y. Since A sc A+ _CB the result follows.

3.1 Corollary

Let G be a subgroup of GL (n, F) and U a unipotent G-hypercyclic normal subgroup of G. Then U has G-paraheight at most +z (n - 1).

3.2 Proof

Let V be the n-row vector space over F regarded as a G-module in the usual way. U is unitriangularizable over F [IO, I.211 and so there exists a non-trivial subspace of V on which U acts trivially. Let W = C, (U).

Then d = dim, W 3 1 and W is G-invariant. By induction on II, UCo(V/W) / Co(V/ W) has G-paraheight at most

+ (n - d) (n - d - 1). By 4.2 U n C, (V/W) has G-paraheight at most d (n - d). Thus, U has G-paraheight at most t (n - d) (n + d - 1) < + n (n -1) since I < d < n.

3.4 LEMMA

If G is an irreducible subgroup of GL (n, F) then X (G) contains a diagonalizable subgroup A normal in G such that /\(G)/A is isomorphic to a subgroup of S, the symmetric group on n letters.

3.5 Proof

Let V be the n-row vector space over F regarded as G-module in the usual way and suppose that {V, ,V,..., V,} is a minimal system of imprimitivity of V as FG-module. If H = NG (Vf), H acts primitively and irreducibly on Vi ([lo] 1.10). Let + be the induced homomorphism of H into AutF (VJ and put 2 = (H n h (G)) + n & (H$). H n A(G) is an H-hypercyclic subgroup of H. If (H n h (G)) + # 2 there exists an element h of (H n X (G)) + \ 2 such that (2, h) is normal in H4. But clearly (2, h) is abelian and so lies in the centre of H$ by Blichfeldt's theorem [10, 1.131. This contradiction shows that Hn X (G) acts on Vi as a group of scalars. Let A = X (G) n h No (VJ.i = l Then A is a diagonalizable normal subgroup of G and /\ (G)/A is isomorphic to a subgroup of S, (since the elements of G permute the Vi among them-selves).

4. Main Theorem

If G is a subgroup of GL (n, F) G has paraheight at most w + [log, n!]. If G is a subgroup of GL (n, R) where R is a finitely generated integral domain then G has finite paraheight.

4.1 Proof

There exist irreducible representations pr, pa ,…, p,. of G and an x in GL (n, F) such that for all g in G. Let U = & ker pi and put H = {diag (gp, ,..., gp7) : g E G > C GL (n, F). U is a unipotent normal subgroup of G and G/U is isomorphic to the completely reducible subgroup H of GL (n, F). By 4.4 H contains a diagonalizable subgroup A that is normal in H such that h (H)/A is isomorphic to a subgroup of S,. Clearly A (H)/A is H-hypercyclic with H-paraheight at most[log, n!]. By [lo] 1.12 (H : C, (A)) is finite. Hence A has H-paraheight at most w (4.1) and thus H has paraheight at most w + [log, n!].

Since G/ U s H, h (G)/ U n h (G) has G-paraheight at most w + [log, n!] and by 4.3 U n A(G) has G-paraheight at most +z (n - 1). Thus, G has paraheight at most w + [logsn!]. Suppose now that G _CGL (n, R). It follows from [IO] 4.10 that A is finitely generated. Therefor by 4.1 H, and thus G, has finite paraheight.

We now consider what paraheights are possible for hypercyclic linear groups of low degree. For each positive integer n and each characteristic p denote by ~ (n, p) the upper bound of the paraheights of hypercyclic linear groups of degree n over a field of characteristic p. We shall see that o (n, p) is never a limit ordinal.

4.2 Proposition

There exist no hypercyclic linear groups with paraheight w.

4.3 Proof

Let G be a hypercyclic subgroup of GL (n, F) with paraheight w. By induction on n and 4.3 G is irreducible. Hence G contains an abelian normal subgroup A of finite index. G contains a series of normal subgroups (1) = G, cG,C ... c G, = G of length w such that every subgroup of Gi + JG, is abelian and normal in G. Since (G : A) is finite there exists a finite r such that AG, = G. Then G/G, is abelian and so G has paraheight at most r + 1. This contradiction proves the proposition. However, there do exist linear groups with paraheight w. These may be constructed exactly as the example 8.7 of [lo] using examples (e.g. (C's, 1 Cs) constructed below in place of C,, 1 C,.

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6. Auhors' Contributions

Behnam Razzaghmaneshi: theorem design, writing, submission and publication.

7. Conflicts of Interest

No conflicts of interest.

8. Ethics Approval

Not applicable.

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